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Non-renormalizability of noncommutative SU(2) gauge theory with fermions

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Abstract: We analyze the divergent part of the one-loop effective action for the noncommutative SU(2) gauge theory coupled to the fermions in the fundamental representation. We show that the divergencies in the 2-point and the 3-point functions in the θ-linear order can be renormalized, while the divergence in the 4-point fermionic function cannot.

Keywords: Renormalization Regularization and Renormalons, Non-Commutative Geometry.

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1. Introduction

Although discussed for quite some time, the question of renormalizability of field theories on noncommutative $\mathbb{R}^4$ has not been settled in a satisfactory way yet. Noncommutativity of the coordinates, i.e., a relation of the type

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}\,, \tag{1.1}$$

puts the lower bound on the coordinate measurements, so one would expect that it also implies a natural ultraviolet cut-off and acts as a regulator. However, this idea has not been successfully implemented in models.

Usually one represents the noncommutative field $\hat{\Phi}(\hat{x})$ by a function $\hat{\Phi}(x)$ on $\mathbb{R}^4$ and encodes the noncommutativity in the multiplication rule ($\star$-product). For example, for constant $\theta^{\mu\nu}$, the field multiplication is given by the Moyal-Weyl product:

$$\phi(x) \star \chi(x) = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}}\phi(x)\chi(y)|_{y\to x}\,. \tag{1.2}$$

This product is nonlocal, and so is the field theory defined by it.

The most extensive study of renormalizability of noncommutative (NC) field theories was done for the scalar field theory [1]. One quantizes perturbatively: the diagrams in the perturbation theory split into planar and nonplanar. The planar diagrams reproduce the behavior of the underlying commutative theory; on the other hand, UV divergencies in the nonplanar graphs get regulated by the effective cut-off $(\theta p)^{-1}$. But for the exceptional

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values of the momenta (when the momentum flow into the loop is zero), these contributions become infinite. The reappearance of the UV divergencies in the infrared sector is related to nonlocality of the theory. Renormalizability of noncommutative Φ^4 has been analyzed in the recent papers [2]–[4] from the point of view of the Wilson-Polchinski RG equation. The renormalization procedure was defined in some special cases; it was shown that this procedure is different from the usual planar renormalization.

Noncommutative gauge theories, in particular U(1) and U(N), have been studied on the similar lines, too. The UV/IR mixing appears in the Feynman graphs in the same way as for the scalar field theory [5, 6], so the question of renormalizability has the same status. However, for the gauge theories there is another representation. As shown by Seiberg and Witten [7], noncommutative and commutative gauge theories are equivalent. This equivalence is realized by a mapping relating the representation (gauge fields, matter fields) of NC gauge symmetry to the fields carrying the representation of its commutative counterpart. The SW map is given as a series in powers of $\theta^{\mu\nu}$. Classical action is also expanded in θ: in the zero-th order it reduces to the action of ordinary gauge theory; additional terms can be treated as couplings. Nonlocality shows up in the infinite number of interactions. This realization gives another framework to address the issue of renormalizability: it in particular makes sense in the limit of small noncommutativity, $\theta \to 0$. One of the features of 'θ-expanded' approach is that arbitrary gauge groups and tensor products can be represented, so noncommutative generalizations of the standard model have been constructed [8].

In this paper, we study the renormalizability of NC SU(2) in the θ-expanded approach. Renormalizability of the θ-expanded NC gauge theories has been addressed in papers [9]–[12] for the gauge group U(1) with or without fermionic matter and for its supersymmetric extension. Here we discuss the SU(2) theory coupled to fermions in the fundamental representation. Although the SU(2) gauge theory technically differs from the abelian U(1) in many details, the conclusion concerning renormalizability is the same. If fermions are massive, theory is not renormalizable. For massless fermions the theory is "almost renormalizable", meaning that there is only one divergent term in the effective action which cannot be absorbed by the SW field redefinition scheme. The method which we used to calculate the divergent contributions is the background field method. As it was thoroughly explained in [12], we skip the technical points here, referring also to the standard literature [13]. The calculations for SU(2) are quite involved in comparison with U(1), so in this paper we discuss only the θ-linear order and find the divergent parts of the 2-point, 3-point and fermionic 4-point functions. Previously, the results for U(1) in the θ-linear order were given in [9, 10]; for the 2-point functions they were extended in [12] to the θ^2-order.

The plan of the paper is the following. In section 2 we describe the classical action for noncommutative SU(2). In the section 3 all necessary steps for the perturbative quantization are done. The results for the divergencies of the 2-point, 3-point and fermionic 4-point functions are given in the sections 4 and 5. The results which are obtained and some further issues concerning renormalizability are discussed in the last section.

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2. The model

The general construction of gauge theories on noncommutative space and their relation to the SW map were introduced in [14]–[16]; we will repeat only a few relevant steps.

The noncommutative space is an algebra generated by a set of noncommuting coordinates $\hat{x}^\mu$; in general they obey relations, $\mathcal{R}(\hat{x}) = 0$; for example (1.1). Physical fields $\hat{\psi}(\hat{x})$ are functions of the coordinates. We want to describe the gauge theory: let $\alpha(x) = \alpha^a(x)T^a$ be a (commutative) gauge parameter and T^a — generators of a Lie group. The field $\hat{\psi}(\hat{x})$ transforms covariantly under the infinitesimal gauge transformation $\hat{\Lambda}_\alpha(\hat{x})$ if $\hat{\delta}_\alpha \hat{\psi}(\hat{x}) = i\hat{\Lambda}_\alpha(\hat{x})\hat{\psi}(\hat{x})$. The gauge degrees of freedom are inner, so the coordinates $\hat{x}^\mu$ are invariant under gauge transformations: $\hat{\delta}_\alpha \hat{x}^\mu = 0$. However, one can define the 'covariant coordinates' $\hat{X}^\mu$ introducing the gauge potentials $\hat{A}_\mu(\hat{x})$: $\hat{X}^\mu = \hat{x}^\mu + \theta^{\mu\nu}\hat{A}_\nu(\hat{x})$. They have the transformation property $\hat{\delta}_\alpha \hat{X}^\mu = i[\hat{\Lambda}_\alpha, \hat{X}^\mu]$, when the vector potential transforms appropriately. In order that the infinitesimal gauge transformations close, one impose

$$\hat{\delta}_\alpha \hat{\delta}_\beta - \hat{\delta}_\beta \hat{\delta}_\alpha = \hat{\delta}_{\alpha\times\beta} \,, \tag{2.1}$$

where $\alpha \times \beta = -i[\alpha, \beta]$ denotes the composition of two transformations.

In principle, for a given set of relations $\mathcal{R} = 0$, the noncommutative coordinates $\hat{x}^\mu$ can be represented by the coordinates x^μ on a commutative manifold, and the multiplication by some $\star$-product. As already mentioned, in the case of the constant noncommutativity (1.1) (which we here consider), the $\star$-product is the Moyal product (1.2); it is possible to construct $\star$-products for the other cases, too. Expanding the gauge parameter and physical fields in θ, the condition (2.1) becomes an equation (Seiberg-Witten) for the coefficients in the expansion. A solution of the SW equation is

$$\begin{aligned}
\hat{\Lambda}(x) &= \alpha(x) + \frac{1}{4}\,\theta^{\mu\nu}\,\{\partial_\mu\alpha(x), A_\nu(x)\} + \cdots \\
\hat{A}_\rho(x) &= A_\rho(x) - \frac{1}{4}\,\theta^{\mu\nu}\,\{A_\mu(x), \partial_\nu A_\rho(x) + F_{\nu\rho}(x)\} + \cdots \\
\hat{\psi}(x) &= \psi(x) - \frac{1}{2}\,\theta^{\mu\nu} A_\mu(x)\partial_\nu\psi(x) + \frac{i}{4}\,\theta^{\mu\nu} A_\mu(x) A_\nu(x)\psi(x) + \cdots ;
\end{aligned} \tag{2.2}$$

the terms of the second order in θ are given in [16]. In the last formula $\alpha(x)$, $A_\mu(x)$ and $\psi(x)$ denote the commutative gauge parameter, the vector potential and the fermionic field:

$$A_\mu = A^a_\mu T^a \,, \qquad D_\mu\psi = \partial_\mu\psi - iA_\mu\psi \,, \tag{2.3}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\,[A_\mu, A_\nu] \tag{2.4}$$

which correspond to the noncommutative quantities. As terms linear in θ in (2.2) contain anticommutators of T^a, it is clear that NC gauge fields $\hat{\Lambda}(x)$, $\hat{A}_\mu(x)$ do not close into the Lie algebra; they take values in the enveloping algebra of the gauge group.

The other property of the solution (2.2) is that it is not unique. It can be shown [9, 17] that, if $A^{(n)}_\mu$, $\psi^{(n)}$ is a solution of the SW equation (2.1) up to the order n in θ, then by adding any gauge covariant expression (of appropriate dimension) with exactly n factors of θ to $A^{(n)}_\mu$, $\psi^{(n)}$ one obtains another solution. This is similar to the relation between the

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solutions of inhomogeneous and the corresponding homogeneous linear equation; in a way, it expresses the nonlocality of the theory. As pointed out in [9], this nonuniqueness can be used to subtract the divergent terms in the effective action and to regularize the theory — one can think of such a procedure as a sort of 'dressing' of the 'bare' fields, that is, the choice of the physical ones.

We now proceed to the action. For NC SU(2) Yang-Mills theory the classical action is given with

$$S = \int d^4x\, \bar{\hat{\psi}} \star (i\gamma^\mu \hat{D}_\mu - m)\hat{\psi} - \frac{1}{4}\int d^4x\, \mathrm{Tr}(\hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu})\,, \tag{2.5}$$

where the noncommutative field strength $\hat{F}_{\mu\nu}$ is

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i(\hat{A}_\mu \star \hat{A}_\nu - \hat{A}_\nu \star \hat{A}_\mu)\,, \tag{2.6}$$

and the covariant derivative

$$\hat{D}_\mu \hat{\psi} = \partial_\mu \hat{\psi} - i\hat{A}_\mu \star \hat{\psi}\,. \tag{2.7}$$

The commutative counterpart $\psi(x)$ of the matter field $\hat{\psi}(x)$ is in the fundamental representation of SU(2), $T^a = \frac{\sigma_a}{2}$. Inserting the expansion (2.2) into (2.5), we get the action in the θ-linear order [16]:

$$\begin{aligned}
S &= S_0 + S_{1,A} + S_{1,\psi}\,, && (2.8)\\
S_0 &= \int d^4x \left(\bar{\psi}\,(i\gamma^\mu D_\mu - m)\,\psi - \frac{1}{4}F^{\mu\nu a}F^a_{\mu\nu} \right), && (2.9)\\
S_{1,A} &= 0\,,\\
S_{1,\psi} &= \frac{1}{2}\theta^{\rho\sigma}\int d^4x \left(-i\bar{\psi}\gamma^\mu F_{\mu\rho} D_\sigma \psi + \frac{1}{2}\bar{\psi}F_{\rho\sigma}(-i\gamma^\mu D_\mu + m)\psi \right) && (2.10)\\
&= \int d^4x \left(-\frac{1}{8}\theta^{\rho\sigma}\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\bar{\psi}F_{\mu\nu}\gamma^\beta(i\partial_\alpha + A_\alpha)\psi + \frac{m}{4}\theta^{\mu\nu}\bar{\psi}F_{\mu\nu}\psi \right).
\end{aligned}$$

In order to simplify the notation we introduced the symbol $\Delta^{\alpha\beta\gamma}_{\sigma\rho\mu}$ defined as

$$\Delta^{\alpha\beta\gamma}_{\sigma\rho\mu} = \delta^\alpha_\sigma \delta^\beta_\rho \delta^\gamma_\mu - \delta^\alpha_\rho \delta^\beta_\sigma \delta^\gamma_\mu + (\text{cyclic } \alpha\beta\gamma) = -\epsilon^{\alpha\beta\gamma\lambda}\epsilon_{\sigma\rho\mu\lambda}\,. \tag{2.11}$$

The bosonic θ-linear term $S_{1,A}$ vanishes (unlike in the U(1) case) because it is proportional to the symmetric coefficients $d^{abc} = \mathrm{Tr}(\{T^a, T^b\}T^c)$, and for SU(2) these coefficients are zero for all irreducible representations.

In the functional integration we will treat $(A_\mu,\ \psi)$ as a multiplet so we want both fields to be real (or to be complex). Therefore we write the Dirac spinor ψ in terms of the Majorana spinors $\psi_{1,2}$. For the charge-conjugated spinor $\psi^C = C\bar{\psi}^T$ Majorana spinors are given by $\psi_{1,2} = \frac{1}{2}(\psi \pm \psi^C)$; vice versa, $\psi = \psi_1 + i\psi_2$.[1] To express the action in terms of Majorana spinors, one has to use explicitly the form of Pauli matrices (i.e., the representation of the group generators). As σ_2 is antisymmetric and σ_1, σ_3 are symmetric matrices, the gauge field A^2_μ couples to Majorana spinors differently from A^1_μ, A^3_μ. The

[1] We encounter some of the identities which Majorana spinors satisfy: $\bar{\phi}\chi = \bar{\chi}\phi$; $\bar{\phi}\gamma_\mu\chi = -\bar{\chi}\gamma_\mu\phi$; $\bar{\phi}\sigma_{\mu\nu}\chi = -\bar{\chi}\sigma_{\mu\nu}\phi$; $\bar{\phi}\gamma_5\chi = \bar{\chi}\gamma_5\phi$; $\bar{\phi}\gamma_\mu\gamma_5\chi = \bar{\chi}\gamma_\mu\gamma_5\phi$.

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action reads:

$$
\begin{aligned}
S_0 = & \int d^4x \left(\bar{\psi}_1 \left(i\partial\!\!\!/ - m + A\!\!\!/^2 \frac{\sigma_2}{2} \right) \psi_1 + \bar{\psi}_2 \left(i\partial\!\!\!/ - m + A\!\!\!/^2 \frac{\sigma_2}{2} \right) \psi_2 + \right. \\
& \left. + i\bar{\psi}_1 \left(A\!\!\!/^1 \frac{\sigma_1}{2} + A\!\!\!/^3 \frac{\sigma_3}{2} \right) \psi_2 - i\bar{\psi}_2 \left(A\!\!\!/^1 \frac{\sigma_1}{2} + A^3 \frac{\sigma_3}{2} \right) \psi_1 - \frac{1}{4} F^{\mu\nu a} F^a_{\mu\nu} \right), \qquad (2.12) \\
S_1 = & -\frac{1}{16} \theta^{\rho\sigma} \Delta^{\mu\nu\alpha}_{\rho\sigma\beta} \int d^4x \left(i\bar{\psi}_1 \gamma^\beta \left(\left(F^1_{\mu\nu} \frac{\sigma_1}{2} + F^3_{\mu\nu} \frac{\sigma_3}{2} \right) \overrightarrow{\partial}_\alpha - \overleftarrow{\partial}_\alpha \left(F^1_{\mu\nu} \frac{\sigma_1}{2} + F^3_{\mu\nu} \frac{\sigma_3}{2} \right) \right) \psi_1 + \right. \\
& + i\bar{\psi}_2 \gamma^\beta \left(\left(F^1_{\mu\nu} \frac{\sigma_1}{2} + F^3_{\mu\nu} \frac{\sigma_1}{2} \right) \overrightarrow{\partial}_\alpha - \overleftarrow{\partial}_\alpha \left(F^1_{\mu\nu} \frac{\sigma_1}{2} + F^3_{\mu\nu} \frac{\sigma_3}{2} \right) \right) \psi_2 - \\
& - \bar{\psi}_1 \gamma^\beta \frac{\sigma_2}{2} (F^2_{\mu\nu} \overrightarrow{\partial}_\alpha - \overleftarrow{\partial}_\alpha F^2_{\mu\nu}) \psi_2 + \\
& + \bar{\psi}_2 \gamma^\beta \frac{\sigma_2}{2} \left(F^2_{\mu\nu} \overrightarrow{\partial}_\alpha - \overleftarrow{\partial}_\alpha F^2_{\mu\nu} \right) \psi_1 + \\
& \left. + \frac{i}{2} F^a_{\mu\nu} A^a_\alpha \left(\bar{\psi}_1 \gamma^\beta \psi_2 - \bar{\psi}_2 \gamma^\beta \psi_1 \right) \right). \qquad (2.13)
\end{aligned}
$$

This will be the initial point for the quantization.

3. One-loop effective action

Background field method is one of the standard methods to obtain divergent and finite quantum contributions to the classical action [13]. In the first step, one expands fields around their classical configuration, i.e. splits the fields into the background (classical) part and the quantum correction:

$$A_\mu \to A_\mu + \mathcal{A}_\mu , \qquad \psi_{1,2} \to \psi_{1,2} + \Psi_{1,2} . \qquad (3.1)$$

Quantum fields are denoted here by $\mathcal{A}_\mu$, $\Psi_{1,2}$. The functional integration over the quantum fields in the generating functional is then performed; the effective action, Γ, is the Legendre transformation of the generating functional. In the saddle-point approximation, the integration gives:

$$\Gamma[A_\mu, \psi_1, \psi_2] = S[A_\mu, \psi_1, \psi_2] + \frac{i}{2} \,\mathrm{Sdet}\, \log S^{(2)}[A_\mu, \psi_1, \psi_2] . \qquad (3.2)$$

Sdet denotes the functional superdeterminant and $S^{(2)}$ is the second functional derivative of the classical action. For polynomial interactions, the second derivative can be obtained from the quadratic part of the action; it is an expression of the type:

$$\int d^4x \,(\mathcal{A}^f_\mu \quad \bar{\Psi}_1 \quad \bar{\Psi}_2) \,\mathcal{B} \begin{pmatrix} \mathcal{A}^e_\nu \\ \Psi_1 \\ \Psi_2 \end{pmatrix}, \qquad (3.3)$$

where we wrote $\mathcal{B}$ instead of $S^{(2)}$ as, in fact, we include the gauge fixing term, too. We use the background field analogue of the Feynman-'t Hooft gauge [13],

$$S_{\mathrm{GF}} = -\frac{1}{2} \int d^4x (D_\mu \mathcal{A}^{\mu a})^2 , \qquad (3.4)$$

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with the gauge fixing parameter equal to 1. D_μ is the background covariant derivative, $D_\mu \mathcal{A}^{\nu\alpha} = \partial_\mu \mathcal{A}^{\nu a} + \epsilon^{abc} A_\mu^b \mathcal{A}^{\nu c}$. To calculate the one-loop correction

$$\Gamma^{(1)} = \frac{i}{2} \log \operatorname{Sdet} \mathcal{B} = \frac{i}{2} \operatorname{STr} \log \mathcal{B}$$

perturbatively, one usually expands $\log \mathcal{B}$. In correspondence with the notation for the fields, $\mathcal{B}$ is a 3×3 block matrix

$$\mathcal{B} = \begin{pmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} & \mathcal{B}_{13} \\ \mathcal{B}_{21} & \mathcal{B}_{22} & \mathcal{B}_{23} \\ \mathcal{B}_{31} & \mathcal{B}_{32} & \mathcal{B}_{33} \end{pmatrix} .$$

The submatrices $\mathcal{B}_{12}$, $\mathcal{B}_{13}$, $\mathcal{B}_{21}$ and $\mathcal{B}_{31}$ are Grassmann-odd while the rest are Grassmann-even; the supertrace is defined by $\operatorname{STr} \mathcal{B} = \operatorname{Tr} \mathcal{B}_{11} - \operatorname{Tr} \mathcal{B}_{22} - \operatorname{Tr} \mathcal{B}_{33}$. $\mathcal{B}$ depends on the classical fields. One should keep in mind that A_μ^a (for $a = 1, 2, 3$) is a triplet, while ψ_1 and ψ_2 are dublets of the SU(2) group.

In the absence of the interaction, $\mathcal{B}$ is the inverse propagator; in general case, one can separate the kinetic part:

$$\mathcal{B} = \begin{pmatrix} \frac{1}{2} g_{\alpha\beta} \delta^{ab} \Box & 0 & 0 \\ 0 & i\not{\partial} & 0 \\ 0 & 0 & i\not{\partial} \end{pmatrix} + \mathcal{M}.$$

We are to expand $\log \mathcal{B}$ around identity $\mathcal{I} = \operatorname{diag}(g_{\mu\nu}\delta^{ab}, 1, 1)$. To achieve this, we multiply $\mathcal{B}$ by $\mathcal{C}$ [18]:

$$\mathcal{C} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -i\not{\partial} & 0 \\ 0 & 0 & -i\not{\partial} \end{pmatrix} ,$$

and then for the one-loop correction we obtain

$$\begin{aligned} \Gamma^{(1)} &= \frac{i}{2} \operatorname{STr} \log(\mathcal{B}\mathcal{C}) + \frac{i}{2} \operatorname{STr} \log \mathcal{C}^{-1} \\ &= \frac{i}{2} \operatorname{STr} \log(\mathcal{I} + \Box^{-1} \mathcal{M}\mathcal{C}) + \frac{i}{2} \operatorname{STr} \log \mathcal{C}^{-1} + \frac{i}{2} \operatorname{STr} \log \Box \,. \end{aligned} \tag{3.5}$$

The second and the third terms, being independent on the fields, can be included in the infinite renormalization. Note that now the propagator for all fields is $\Box^{-1}$. The operator $\Box^{-1}\mathcal{M}\mathcal{C}$ defines the rules in the perturbation expansion.

To get the structure of the expansion more clearly, we decompose $\mathcal{M}\mathcal{C}$ into the sum

$$\mathcal{M}\mathcal{C} = N_0 + N_1 + N_2 + T_1 + T_2 + T_3 \tag{3.6}$$

with respect to the number of fields — indices denote their number in a given term. N_0, N_1 and N_2 originate from the commutative theory, while T_1, T_2 and T_3 are the noncommutative interactions linear in θ. One can read $N_0 \ldots T_3$ from the action (2.12) and (2.13), after the separation of the part quadratic in the quantum fields.

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In the θ^0 order we get

$$N_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & im\not{\partial} & 0 \\ 0 & 0 & im\not{\partial} \end{pmatrix} . \tag{3.7}$$

N_1 is lengthy. If we write it as

$$N_1 = \begin{pmatrix} N & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & -u_{23} & u_{22} \end{pmatrix} , \tag{3.8}$$

its matrix elements are given by

$$\begin{aligned}
N^{fe}_{\mu\nu} &= 2\epsilon^{efb}(\partial_\mu A^b_\nu - \partial_\nu A^b_\mu) + \epsilon^{efb} g_{\mu\nu}(A^{\alpha b}\overrightarrow{\partial}_\alpha - \overleftarrow{\partial}_\alpha A^{\alpha b}) \\
u_{12} &= \left(-\bar{\psi}_2\frac{\sigma_1}{2} \quad -i\bar{\psi}_1\frac{\sigma_2}{2} \quad -\bar{\psi}_2\frac{\sigma_3}{2}\right)^T \gamma^\mu \not{\partial} \\
u_{13} &= \left(\bar{\psi}_1\frac{\sigma_1}{2} \quad -i\bar{\psi}_2\frac{\sigma_2}{2} \quad \bar{\psi}_1\frac{\sigma_3}{2}\right)^T \gamma^\mu \not{\partial} \\
u_{21} &= 2\gamma^\nu \left(i\frac{\sigma_1}{2}\psi_2 \quad \frac{\sigma_2}{2}\psi_1 \quad i\frac{\sigma_3}{2}\psi_2\right) \\
u_{31} &= 2\gamma^\nu \left(-i\frac{\sigma_1}{2}\psi_1 \quad \frac{\sigma_2}{2}\psi_2 \quad -i\frac{\sigma_3}{2}\psi_1\right) \\
u_{22} &= -i\not{A}^2\frac{\sigma_2}{2}\not{\partial} \\
u_{23} &= \left(\not{A}^1\frac{\sigma_1}{2} + \not{A}^3\frac{\sigma_3}{2}\right)\not{\partial} .
\end{aligned}$$

The remaining, N_2, is

$$N_2 = \begin{pmatrix} M & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{3.9}$$

with

$$M^{fe}_{\mu\nu} = -g_{\mu\nu}\delta^{ef} A^a_\alpha A^{\alpha a} + g_{\mu\nu} A^e_\alpha A^{f\alpha} + 2A^e_\mu A^f_\nu - 2A^f_\mu A^e_\nu .$$

We will introduce T_1, T_2 and T_3 symbolically; the full expressions are given in the Appendix. They are of the form

$$T_1 = \begin{pmatrix} 0 & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & -p_{23} & p_{22} \end{pmatrix} , \tag{3.10}$$

$$T_2 = \begin{pmatrix} Q & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & -q_{23} & q_{22} \end{pmatrix} , \qquad T_3 = \begin{pmatrix} R & r_{12} & r_{13} \\ r_{21} & 0 & r_{23} \\ r_{31} & -r_{23} & 0 \end{pmatrix} . \tag{3.11}$$

As already stressed, the representation of SU(2) is not real (symmetric), and the interaction of the gauge fields with the Majorana spinors cannot be written in a manifestly covariant way. But all given operators have the same structure which is the consequence of the fact that the action was originally in Dirac spinors. Note that for the massless fermions $N_0 = 0$; N_1, T_1 and T_2 drastically simplify, as well.

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4. Divergencies, 2-point functions

Introducing the decomposition (3.6), for the one-loop correction (3.5) we get

$$\begin{aligned}\Gamma^{(1)} &= \frac{i}{2}\,\mathrm{STr}\log\left(\mathcal{I}+\Box^{-1}\mathcal{M}\mathcal{C}\right) \qquad (4.1)\\ &= \frac{i}{2}\sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n}\,\mathrm{STr}\left(\Box^{-1}N_0+\Box^{-1}N_1+\Box^{-1}N_2+\Box^{-1}T_1+\Box^{-1}T_2+\Box^{-1}T_3\right)^n .\end{aligned}$$

Our notation allows to extract the contributions from different terms in the expansion (4.1) easily. Parts of the effective action which give the 2-point functions have two classical fields. This means that the sum of indices in the monomials which we are interested in is equal to 2. Since there is an operator with the index 0, in principle, infinitely many terms contribute to the n-point functions. However, as we are calculating the divergencies only, we will need just finite number of terms: $\Box^{-1}N_0$ behaves as p^{-1} in the momentum space, and the integrals become convergent for $(\Box^{-1}N_0)^k$ of a high enough degree.

For the 2-point function in the zero-th order, power counting gives that the traces which contribute are: $\mathrm{STr}(\Box^{-1}N_1\Box^{-1}N_1)$, $\mathrm{STr}(\Box^{-1}N_0\Box^{-1}N_2)$ and $\mathrm{STr}(\Box^{-1}N_0(\Box^{-1}N_1)^2)$. Performing the Fourier transformation and the dimensional regularization we obtain

$$\begin{aligned}\mathrm{STr}\left(\Box^{-1}N_1\Box^{-1}N_1\right) &= \frac{i}{(4\pi)^2\epsilon}\int d^4x\,\left(-24A^a_\mu\Box A^{\mu a}-24(\partial_\mu A^{\mu a})^2+6i\bar\psi\partial\!\!\!/\psi\right),\\ \mathrm{STr}\left(\Box^{-1}N_0\Box^{-1}N_2\right) &= 0\,,\\ \mathrm{STr}\left(\Box^{-1}N_0(\Box^{-1}N_1)^2\right) &= \frac{i}{(4\pi)^2\epsilon}\int d^4x\,12m\bar\psi\psi\,.\end{aligned}$$

Taken together, these terms give the standard result of the commutative theory

$$\begin{aligned}\Gamma_{2A,2\psi} &= -\frac{i}{4}\,\mathrm{STr}(\Box^{-1}N_1\Box^{-1}N_1)+\frac{i}{2}\,\mathrm{STr}(\Box^{-1}N_0(\Box^{-1}N_1)^2)\\ &= \frac{1}{(4\pi)^2\epsilon}\int d^4x\,\left(-6A^a_\mu\Box A^{\mu a}-6(\partial_\mu A^{\mu a})^2+\frac{3i}{2}\bar\psi\partial\!\!\!/\psi-6m\bar\psi\psi\right), \qquad (4.2)\end{aligned}$$

to which the contribution of the ghost action is to be added. Since the gauge fixing term (3.4) does not depend on the noncommutativity parameter θ, the result will be the usual one. The ghost action following from (3.4) is

$$S_{gh} = -\int d^4x\,\bar c^a(D^2\delta^{ad}+\epsilon^{abd}D^\mu\mathcal{A}^b_\mu)c^d\,. \qquad (4.3)$$

Integration over the ghost fields c^a and $\bar c^a$ gives the ghost contribution to the one-loop effective action [19, 13]

$$\Gamma_{2,gh} = \frac{1}{(4\pi)^2\epsilon}\,\frac{1}{3}\int d^4x\,F^a_{\mu\nu}F^{\mu\nu a}\,. \qquad (4.4)$$

Now we consider the θ-linear order. Potentially divergent terms are

$$\mathrm{STr}\left(\Box^{-1}N_1\Box^{-1}T_1\right),\quad \mathrm{STr}\left(\left(\Box^{-1}N_0\Box^{-1}N_1+\Box^{-1}N_1\Box^{-1}N_0\right)\Box^{-1}T_1\right),\quad \mathrm{STr}\left(\Box^{-1}N_0\Box^{-1}T_2\right)$$

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and

$$\mathrm{STr}\left(\left(\left(\Box^{-1}N_0\right)^2\Box^{-1}N_1+\Box^{-1}N_0\Box^{-1}N_1\Box^{-1}N_0+\Box^{-1}N_1\left(\Box^{-1}N_0\right)^2\right)\Box^{-1}T_1\right).$$

The dimensional regularization gives the following:

$$\begin{aligned}
\mathrm{STr}(\Box^{-1}N_1\Box^{-1}T_1) &= \frac{i}{4(4\pi)^2\epsilon}\theta^{\mu\nu}\times\\
&\quad\times\int d^4x\,\left(i\epsilon_{\mu\nu\rho\sigma}\bar\psi\gamma_5\gamma^\sigma\Box\partial^\rho\psi+2m\bar\psi\sigma_{\nu\alpha}\partial_\mu\partial^\alpha\psi-m\bar\psi\sigma_{\mu\nu}\Box\psi\right),\\
\mathrm{STr}\left((\Box^{-1}N_0\Box^{-1}N_1+\Box^{-1}N_1\Box^{-1}N_0)\Box^{-1}T_1\right) &= 0\,,\\
\mathrm{STr}(\Box^{-1}N_0\Box^{-1}T_2) &= 0\,,
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{STr}\left(((\Box^{-1}N_0)^2\Box^{-1}N_1+\Box^{-1}N_0\Box^{-1}N_1\Box^{-1}N_0+\Box^{-1}N_1(\Box^{-1}N_0)^2)\Box^{-1}T_1\right)=\\
&\quad=\frac{i}{(4\pi)^2\epsilon}\,\frac{3}{4}\,\theta^{\mu\nu}\int d^4x\,\left(-im^2\epsilon_{\mu\nu\alpha\beta}\bar\psi\gamma_5\gamma^\beta\partial^\alpha\psi+m^3\bar\psi\psi\sigma_{\nu\mu}\psi\right).
\end{aligned}$$

All together, the divergent part of the 2-point function in the linear order reads:

$$\begin{aligned}
\Gamma'_{2\psi} &= \frac{1}{(4\pi)^2\epsilon}\,\frac{1}{8}\,\theta^{\mu\nu}\int d^4x\,\left(i\epsilon_{\mu\nu\rho\sigma}\bar\psi\gamma_5\gamma^\sigma\Box\partial^\rho\psi+2m\bar\psi\sigma_{\nu\alpha}\partial_\mu\partial^\alpha\psi-m\bar\psi\sigma_{\mu\nu}\Box\psi\right)+\\
&\quad+\frac{1}{(4\pi)^2\epsilon}\,\frac{3}{8}\,\theta^{\mu\nu}\int d^4x\,\left(-im^2\epsilon_{\mu\nu\alpha\beta}\bar\psi\gamma_5\gamma^\beta\partial^\alpha\psi+m^3\bar\psi\sigma_{\nu\mu}\psi\right).
\end{aligned}\tag{4.5}$$

This is a nice result: comparing (4.5) with the θ-linear correction for the 2-point functions in NC QED [12], we see that in the SU(2) case also, only fermionic propagator gets a correction; the correction is, up to a factor, the same as the one for U(1). This has further consequences. In NC QED we argued that the massive terms obstruct renormalization, as only for the case $m=0$ one can redefine fields in such a way that the divergent terms disappear. The analysis can be repeated for SU(2) without change. Hence, we come to the conclusion: the NC gauge theories with massive fermions are not renormalizable.

For this reason in the calculations of 3-point and 4-point functions we focus to the massless case. One might add that the calculations become so cumbersome that otherwise they would hardly be doable.

5. 3-point and 4-point functions

The background field method is a gauge covariant method and therefore it gives the covariant results. On the other hand, the separation into 2-point, 3-point etc. functions breaks the gauge covariance: for instance, $\Gamma'_{2\psi}$ given by the formula (4.5) is not covariant; it is a part of the covariant expression (written assuming that $m=0$) which is, up to the order of the covariant derivatives, equal to:

$$\Gamma'_2=\frac{1}{(4\pi)^2\epsilon}\,\frac{i}{8}\,\theta^{\mu\nu}\epsilon_{\mu\nu\rho\sigma}\int d^4x\,\bar\psi\gamma_5\gamma^\sigma D^2D^\rho\psi\,.\tag{5.1}$$

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Writing Γ_2' in the form (5.1), we included the parts of the 3-point, 4-point and 5-point functions. When one calculates the 3-point functions, from the dimensional analysis it is easy to see that, apart from the terms residual from (5.1), basically only two terms contribute (up to the partial integration and various combinations of indices). They are the leading terms in the covariant expressions

$$\theta\bar{\psi}\gamma F(D\psi) \qquad \text{and} \qquad \theta\bar{\psi}\gamma(DF)\psi\,, \tag{5.2}$$

(γ stands for the products of the γ-matrices). However, as we stressed already, the calculation i.e. the organization of terms becomes increasingly difficult, so in order to find the 3-point functions we use a trick. We calculate the coefficients of the terms (5.2) in the 4-point functions. When we are doing this, we can assume that the background spinor field is constant, so the covariant derivative $D_\mu\psi$ reduces to $A_\mu\psi$; in this case $N_1\dots T_3$ also simplify. The gauge covariance enables us to recover the result for the 3-point functions uniquely at the end of the calculation.

Divergent parts of the 3-point functions are in the terms $\mathrm{STr}((\Box^{-1}N_1)^2\Box^{-1}T_1)$ and $\mathrm{STr}(\Box^{-1}N_1\Box^{-1}T_2)$; the corresponding traces in the 4-point functions are $\mathrm{STr}(\Box^{-1}N_1\Box^{-1}T_3)$ and $\mathrm{STr}((\Box^{-1}N_1)^2\Box^{-1}T_2)$. The divergent part of the first trace is

$$\mathrm{STr}(\Box^{-1}N_1\Box^{-1}T_3) = -\frac{i}{(4\pi)^2\epsilon}\,\theta^{\mu\nu}\int d^4x\,\left(6A_\nu^aF_{\mu\alpha}^a\bar{\psi}\gamma^\alpha\psi - 3A_\alpha^aF_{\mu\nu}^a\bar{\psi}\gamma^\alpha\psi\right), \tag{5.3}$$

while for the second one we get

$$\begin{aligned}
\mathrm{STr}((\Box^{-1}N_1)^2\Box^{-1}T_2) \;=\; & \frac{i}{(4\pi)^2\epsilon}\,\theta^{\mu\nu}\times\\
& \times\int d^4x\Bigg(\frac{33}{24}\,A_\mu^a\,(\partial_\nu A_\alpha^a-\partial_\alpha A_\nu^a)\,\bar{\psi}\gamma^\alpha\psi\,-\\
& \qquad -\frac{5}{16}\,A_\alpha^a\,\left(\partial_\nu A_\mu^a-\partial_\mu A_\nu^a\right)\bar{\psi}\gamma^\alpha\psi\,+\\
& \qquad +\frac{1}{2}A^{a\alpha}\,(\partial_\nu A_\alpha^a-\partial_\alpha A_\nu^a)\,\bar{\psi}\gamma_\mu\psi\,+\\
& \qquad +\epsilon^{abc}\Bigg(-\frac{5}{6}\,\epsilon_{\nu\alpha\beta\rho}A^{a\alpha}\left(\partial_\mu A^{b\beta}-\partial^\beta A_\mu^b\right)\times\\
& \qquad\qquad \times\bar{\psi}\frac{\sigma_c}{2}\,\gamma_5\gamma^\rho\psi-\frac{25}{24}\,\epsilon_{\nu\alpha\beta\rho}A_\mu^a\times\\
& \qquad\qquad \times(\partial^\alpha A^{c\beta}-\partial^\beta A^{c\alpha})\bar{\psi}\frac{\sigma_b}{2}\,\gamma_5\gamma^\rho\psi\,-\\
& \qquad\qquad -\frac{1}{16}\,\epsilon_{\nu\mu\alpha\beta}A_\sigma^c(\partial^\beta A^{a\alpha}-\partial^\alpha A^{a\beta})\bar{\psi}\frac{\sigma_b}{2}\,\gamma_5\gamma^\sigma\psi\,-\\
& \qquad\qquad -\frac{1}{8}\,\epsilon_{\nu\mu\alpha\beta}A^{b\alpha}\partial_\sigma A^{a\sigma}\bar{\psi}\frac{\sigma_c}{2}\,\gamma_5\gamma^\beta\psi\Bigg)\Bigg)\,.
\end{aligned} \tag{5.4}$$

When we add these expressions and try to 'covariantize' the result, we obtain a piece which does not match: it is precisely the part of the 2-point function (5.1). The rest gives

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the 3-point function in its covariant form:

$$\Gamma_3' = -\frac{1}{(4\pi)^2\epsilon}\,\frac{1}{2}\,\theta^{\mu\nu}\int d^4x \Big(-\frac{111i}{6}\,\bar\psi\gamma^\alpha F_{\nu\alpha}D_\mu\psi - \\ -\frac{43i}{4}\,\bar\psi\gamma^\alpha F_{\mu\nu}D_\alpha\psi - \frac{3i}{4}\,\bar\psi\gamma^\alpha(D_\alpha F_{\mu\nu})\psi + \\ +2i\bar\psi\gamma_\mu F_{\nu\alpha}D^\alpha\psi + i\bar\psi\gamma_\mu(D^\alpha F_{\nu\alpha})\psi + \\ +\frac{5}{8}\,\epsilon_{\nu\alpha\beta\rho}\bar\psi\gamma_5\gamma^\rho(D_\mu F^{\alpha\beta})\psi - \frac{1}{16}\,\epsilon_{\mu\nu\alpha\beta}\bar\psi\gamma_5\gamma^\sigma(D_\sigma F^{\alpha\beta})\psi + \\ +\frac{1}{8}\,\epsilon_{\mu\nu\alpha\beta}\Big(2\bar\psi\gamma_5\gamma^\beta F^{\rho\alpha}D_\rho\psi + \bar\psi\gamma_5\gamma^\beta(D_\rho F^{\rho\alpha})\psi\Big)\Big)\,. \tag{5.5}$$

4-fermionic vertex has a very important role in the discussion of renormalizability. The corresponding 4-point function is relatively easy to find: to this end one can put $A_\mu^a = 0$ in $N_1 \dots T_3$. The divergent part comes from $\mathrm{STr}((\Box^{-1}N_1)^2\Box^{-1}T_2)$ and $\mathrm{STr}((\Box^{-1}N_1)^3\Box^{-1}T_1)$; the final result is

$$\Gamma_{4\psi}' = \frac{1}{(4\pi)^2\epsilon}\,\frac{9}{32}\,\theta^{\mu\nu}\epsilon_{\mu\nu\rho\sigma}\int d^4x\,\bar\psi\gamma_5\gamma^\sigma\psi\,\bar\psi\gamma^\rho\psi\,. \tag{5.6}$$

(5.1), (5.5) and (5.6) are the main results of our calculation.

6. Discussion

As we mentioned in the introduction, there is no a priori criterion which would fix the nonuniqueness in the SW map. The redefinition of fields allowed by it changes the action; the terms which appear are of the forms:

$$\Delta S_A^{(n)} = \int d^4x\,(D_\nu F^{\mu\nu})\mathbf{A}_\mu^{(n)}\,, \tag{6.1}$$

$$\Delta S_\psi^{(n)} = \int d^4x\,\Big(\bar\psi i\not{D}\mathbf{\Psi}^{(n)} + \bar{\mathbf{\Psi}}^{(n)}i\not{D}\psi\Big)\,, \tag{6.2}$$

written for the case of massless fermions. $\mathbf{A}_\mu^{(n)}$ and $\mathbf{\Psi}^{(n)}$ are gauge covariant expressions of the n-th order in θ. The important thing in (6.1) and (6.2) is that, besides $F^{\mu\nu}$ and ψ, they contain at least one derivative.

The divergencies (5.1), (5.5), (5.6) which we obtained are such that they cannot be subtracted by the usual counterterms. However, if they were of the types (6.1) and (6.2), one could include them in the field redefinitions; thus the theory would be renormalizable in a generalized sense. Analyzing the divergencies, we see that the situation with the NC SU(2) is pretty much the same as with the electrodynamics. We already noted that the propagator correction (5.1) breaks the renormalizability, unless $m = 0$. The 3-point functions in the massless case present no problem, too. Gluon 3- and 4-vertices get no quantum correction in the θ-linear order — there is no classical gluon vertex in (2.13) in that order, either. In this respect the behavior is again similar to U(1), where θ-linear 3-photon vertices did exist: quantum one-loop corrections were precisely of the same form as the corresponding classical vertices [10]. Further, the fermion-gluon 3-vertex (5.5) is

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already written in the form (6.2) with

$$\mathbf{\Psi}^{(1)} = \theta^{\mu\nu} \left(\kappa_1 F_{\mu\nu} + i\kappa_2 \sigma_{\mu\rho} F_\nu{}^\rho + i\kappa_3 \epsilon_{\mu\nu\rho\sigma}\gamma_5 F^{\rho\sigma} + \kappa_4 \sigma_{\mu\nu} D^2\right)\psi\,. \tag{6.3}$$

We observe that the fermions in the renormalized theory would be redefined via the gluon fields, i.e., noncommutativity would be mixed with (or partly immersed into) the gauge interactions.

However, there is a divergent term which spoils the renormalizability: the 4-point function (5.6). It predicts the current-current interaction, and there is no simple way to circumvent this coupling induced by noncommutativity.

In fact, from the dimensional analysis we can see that in the massless case, propagators in NC gauge theories are renormalizable to all orders. Namely, for gauge bosons the n-th order corrections are of the form

$$\underbrace{\theta\ldots\theta}_{n}\, A\, \underbrace{\partial\ldots\partial}_{k}\, A\,, \tag{6.4}$$

while for fermions they are

$$\underbrace{\theta\ldots\theta}_{n}\, \bar\psi\gamma\, \underbrace{\partial\ldots\partial}_{k}\, \psi\,. \tag{6.5}$$

The power counting gives the number of derivatives k: for the gluon propagator $k = 2{+}2n$; for the fermions, $k = 1{+}2n$. This shows that one can, in all orders, transform (6.4) and (6.5) into the desired forms (6.1) and (6.2). The use of the background field method guarantees the gauge covariance.

On the other hand, the vertices are potentially problematic. From the power counting we see that in the linear order the 'wrong' vertex could be

$$\theta(\bar\psi\gamma\psi)^2\,, \tag{6.6}$$

while in the quadratic order we could have, e.g.,

$$\theta^2 F^4\,, \qquad \theta^2(\bar\psi\gamma\psi)^2 F\,. \tag{6.7}$$

These terms contain no derivatives and therefore break the SW generalized renormalization scheme. The term (6.6) is present for both U(1) and SU(2) theories coupled to fermions. An interesting fact, however, is that in both theories it has the same form (a different coefficient), namely

$$\theta^{\mu\nu}\epsilon_{\mu\nu\rho\sigma}\bar\psi\gamma_5\gamma^\sigma\psi\,\bar\psi\gamma^\rho\psi\,, \tag{6.8}$$

whereas the other combinations allowed by covariance, as, e.g. $\theta^{\mu\nu}\epsilon_{\mu\nu\rho\sigma}\bar\psi\gamma_5\gamma^\sigma\frac{\sigma_a}{2}\psi\,\bar\psi\gamma^\rho\frac{\sigma_a}{2}\psi$ or $\theta^{\mu\nu}\bar\psi\gamma_5\gamma_\mu\psi\,\bar\psi\gamma_\nu\psi$, never show up. This opens the possibility that this divergence might cancel in a gauge theory based on the product of gauge groups.

However, we are not in favor of a theory which needs too much fine tuning. Thus we are inclined to interpret our results (and the previous ones, [9, 10, 12]) as an indication that NC gauge theories coupled to fermions are not renormalizable. But before a definite conclusion, one should certainly check whether the specific θ^2-corrections, as $4A$ vertex (6.7), vanish. The presence of the θ^2F^4 divergence would prove non-renormalizability, possibly even for the pure gauge theories. It would also be interesting to understand if there is some systematics in the behavior of various divergent terms.

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A. θ-linear matrix elements

We present here the operators from the expansion (3.6) which are induced by the θ-linear interaction terms. The matrix T_1 containing one background field is

$$T_1 = \begin{pmatrix} 0 & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & -p_{23} & p_{22} \end{pmatrix}, \tag{A.1}$$

where the matrices p_{ij} are given by

$$\begin{aligned}
p_{12} &= -\frac{i}{4}\,\theta^{\rho\sigma}\begin{pmatrix} -m\delta^\nu_\sigma\partial_\rho\bar\psi_1\sigma_1 + \frac{i}{2}\,\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}(\partial_\mu\bar\psi_1)\sigma_1\gamma^\beta\partial_\alpha \\ i\delta^\nu_\sigma m\partial_\rho\bar\psi_2\sigma_2 + \frac{1}{2}\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}(\partial_\mu\bar\psi_2)\sigma_2\gamma^\beta\partial_\alpha \\ -m\delta^\nu_\sigma\partial_\rho\bar\psi_1\sigma_3 + \frac{i}{2}\,\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}(\partial_\mu\bar\psi_1)\sigma_3\gamma^\beta\partial_\alpha \end{pmatrix}\not\partial\,, \\
p_{13} &= -\frac{i}{4}\,\theta^{\rho\sigma}\begin{pmatrix} -m\delta^\nu_\sigma\partial_\rho\bar\psi_2\sigma_1 + \frac{i}{2}\,\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}(\partial_\mu\bar\psi_2)\sigma_1\gamma^\beta\partial_\alpha \\ -im\delta^\nu_\sigma\partial_\rho\bar\psi_1\sigma_2 - \frac{1}{2}\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}(\partial_\mu\bar\psi_1)\sigma_2\gamma^\beta\partial_\alpha \\ -m\delta^\nu_\sigma\partial_\rho\bar\psi_2\sigma_3 + \frac{i}{2}\,\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}(\partial_\mu\bar\psi_2)\sigma_3\gamma^\beta\partial_\alpha \end{pmatrix}\not\partial\,, \\
p_{21} &= \frac{1}{2}\,\theta^{\rho\sigma}\begin{pmatrix} m\delta^\nu_\sigma\sigma_1\psi_1\partial_\rho - \frac{i}{2}\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\gamma^\beta\sigma_1(\partial_a\psi_1)\partial_\mu \\ im\delta^\nu_\sigma\sigma_2\psi_2\partial_\rho + \frac{1}{2}\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\gamma^\beta\sigma_2(\partial_a\psi_2)\partial_\mu \\ m\delta^\nu_\sigma\sigma_3\psi_1\partial_\rho - \frac{i}{2}\,\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\gamma^\beta\sigma_3(\partial_a\psi_1)\partial_\mu \end{pmatrix}^T \\
p_{31} &= \frac{1}{2}\,\theta^{\rho\sigma}\begin{pmatrix} m\delta^\nu_\sigma\sigma_1\psi_2\partial_\rho - \frac{i}{2}\,\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\gamma^\beta\sigma_1(\partial_a\psi_2)\partial_\mu \\ -im\delta^\nu_\sigma\sigma_2\psi_1\partial_\rho - \frac{1}{2}\,\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\gamma^\beta\sigma_2(\partial_a\psi_1)\partial_\mu \\ m\delta^\nu_\sigma\sigma_3\psi_2\partial_\rho - \frac{i}{2}\,\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\gamma^\beta\sigma_3(\partial_a\psi_2)\partial_\mu \end{pmatrix}^T \\
p_{22} &= -\frac{i}{4}\,\theta^{\rho\sigma}\Big(m((\partial_\rho A^1_\sigma)\sigma_1 + (\partial_\rho A^3_\sigma)\sigma_3) - \frac{i}{2}\,\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\gamma^\beta((\partial_\mu A^1_\nu)\sigma_1 + (\partial_\mu A^3_\nu)\sigma_3)\partial_\alpha\Big)\not\partial \\
p_{23} &= \frac{1}{4}\,\theta^{\rho\sigma}\Big(-m(\partial_\rho A^2_\sigma)\sigma_2 - \frac{i}{2}\,\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\gamma^\beta(\partial_\mu A^2_\nu)\sigma_2\partial_\alpha\Big)\not\partial\,.
\end{aligned} \tag{A.2}$$

The matrix T_2 is

$$T_2 = \begin{pmatrix} Q & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & -q_{23} & q_{22} \end{pmatrix} \tag{A.3}$$

with $Q = \begin{pmatrix} a & b & c \\ -b & a & d \\ -c & -d & a \end{pmatrix}$, and

$$\begin{aligned}
a &= \frac{1}{16}\,\theta^{\rho\sigma}\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\left(\partial_\alpha(\bar\psi\gamma^\beta\psi) + \bar\psi\gamma^\beta\psi\partial_\alpha\right) \\
b &= \frac{m}{4}\,\theta^{\mu\nu}(\bar\psi_1\sigma_3\psi_1 + \bar\psi_2\sigma_3\psi_2) + \frac{i}{8}\,\theta^{\rho\sigma}\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\left((\partial_\alpha\bar\psi_1)\gamma^\beta\sigma_3\psi_1 + (\partial_\alpha\bar\psi_2)\gamma^\beta\sigma_3\psi_2\right) \\
c &= -\frac{im}{4}\,\theta^{\mu\nu}(\bar\psi_1\sigma_2\psi_1 - \bar\psi_2\sigma_2\psi_2) - \frac{1}{8}\,\theta^{\rho\sigma}\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\left((\partial_\alpha\bar\psi_1)\gamma^\beta\sigma_2\psi_1 + (\partial_\alpha\bar\psi_2)\gamma^\beta\sigma_2\psi_2\right) \\
d &= \frac{m}{4}\,\theta^{\mu\nu}(\bar\psi_1\sigma_1\psi_1 + \bar\psi_2\sigma_1\psi_2) + \frac{i}{8}\,\theta^{\rho\sigma}\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\left((\partial_\alpha\bar\psi_1)\gamma^\beta\sigma_1\psi_1 + (\partial_\alpha\bar\psi_2)\gamma^\beta\sigma_1\psi_2\right),
\end{aligned}$$

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while the other matrix elements are

$$
\begin{aligned}
q_{12} &= -\frac{i}{8}\,\theta^{\rho\sigma}\begin{pmatrix} -m\delta^{\nu}_{\sigma}(\bar\psi_1 A^2_\rho\sigma_3 + i\bar\psi_2 A^3_\rho\sigma_2) - \frac{i}{2}\,\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}((\partial_\mu A^1_\alpha)\bar\psi_2 + \partial_\mu A^1_\alpha\bar\psi_2)\gamma^\beta \\ -im\delta^{\nu}_{\sigma}(-i\bar\psi_1 A^3_\rho\sigma_1 + i\bar\psi_1 A^1_\rho\sigma_3) - \frac{i}{2}\,\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}((\partial_\mu A^2_\alpha)\bar\psi_2 + \partial_\mu A^2_\alpha\bar\psi_2)\gamma^\beta \\ -m\delta^{\nu}_{\sigma}(-\bar\psi_1 A^2_\rho\sigma_1 - i\bar\psi_2 A^1_\rho\sigma_2) - \frac{i}{2}\,\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}((\partial_\mu A^3_\alpha)\bar\psi_2 + \partial_\mu A^3_\alpha\bar\psi_2)\gamma^\beta \end{pmatrix}\not{\partial} - \\
&\quad - \frac{i}{16}\,\theta^{\rho\sigma}\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\begin{pmatrix} -iA^2_\mu((\partial_\alpha\bar\psi_1) - \bar\psi_1\partial_\alpha)\sigma_3) + A^3_\mu((\partial_\alpha\bar\psi_2) - \bar\psi_2\partial_\alpha)\sigma_2) \\ iA^3_\mu((\partial_\alpha\bar\psi_1) - \bar\psi_1\partial_\alpha)\sigma_1 + iA^1_\mu((\partial_\alpha\bar\psi_1) - \bar\psi_1\partial_\alpha)\sigma_3) \\ iA^2_\mu((\partial_\alpha\bar\psi_1) - \bar\psi_1\partial_\alpha)\sigma_1 - A^1_\mu((\partial_\alpha\bar\psi_2) - \bar\psi_2\partial_\alpha)\sigma_2) \end{pmatrix}\gamma^\beta\not{\partial} \\
q_{13} &= q_{12}|_{\bar\psi_1\to\bar\psi_2,\ \bar\psi_2\to-\bar\psi_1} \\
q_{21} &= \frac{1}{4}\,\theta^{\rho\sigma}\begin{pmatrix} m\delta^{\nu}_{\sigma}(-A^2_\rho\sigma_3\psi_1 + iA^3_\rho\sigma_2\psi_2) + \frac{i}{2}\,\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\gamma^\beta\psi_2((\partial_\mu A^1_\alpha) - A^1_\alpha\partial_\mu) \\ m\delta^{\nu}_{\sigma}(-A^3_\rho\sigma_1\psi_1 + A^1_\rho\sigma_3\psi_2) + \frac{i}{2}\,\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\gamma^\beta\psi_2((\partial_\mu A^2_\alpha) - A^2_\alpha\partial_\mu) \\ m\delta^{\nu}_{\sigma}(-A^2_\rho\sigma_1\psi_1 - iA^1_\rho\sigma_2\psi_2) + \frac{i}{2}\,\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\gamma^\beta\psi_2((\partial_\mu A^3_\alpha) - A^3_\alpha\partial_\mu) \end{pmatrix}^T + \\
&\quad + \frac{1}{8}\,\theta^{\rho\sigma}\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\gamma^\beta\begin{pmatrix} i\sigma_3(\partial_\alpha\psi_1 + (\partial_\alpha\psi_1))A^2_\mu + \sigma_2(\partial_\alpha\psi_2 + (\partial_\alpha\psi_2))A^3_\mu \\ i\sigma_1(\partial_\alpha\psi_1 + (\partial_\alpha\psi_1))A^3_\mu - i\sigma_3(\partial_\alpha\psi_1 + (\partial_\alpha\psi_1))A^1_\mu \\ -i\sigma_1(\partial_\alpha\psi_1 + (\partial_\alpha\psi_1))A^2_\mu - \sigma_2(\partial_\alpha\psi_2 + (\partial_\alpha\psi_2))A^1_\mu \end{pmatrix}^T \\
q_{23} &= q_{21}|_{\psi_1\to\psi_2,\ \psi_2\to-\psi_1} \\
q_{22} &= -i\frac{m}{4}\,\theta^{\rho\sigma}(A^2_\rho A^3_\sigma\sigma_1 + A^1_\rho A^2_\sigma\sigma_3)\not{\partial} - \\
&\quad - \frac{1}{16}\,\theta^{\rho\sigma}\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\gamma^\beta\Big[\sigma_1(\partial_\alpha A^2_\mu A^3_\nu + A^2_\mu A^3_\nu\partial_\alpha) + \sigma_3(\partial_\alpha A^1_\mu A^2_\nu + A^1_\mu A^2_\nu\partial_\alpha)\Big]\not{\partial} \\
q_{23} &= \frac{m}{4}\,\theta^{\rho\sigma}A^3_\rho A^1_\sigma\sigma_2\not{\partial} - \frac{i}{16}\,\theta^{\rho\sigma}\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\gamma^\beta\sigma_2(\partial_\alpha A^3_\mu A^1_\nu + A^3_\mu A^1_\nu\partial_\alpha)\not{\partial} - \\
&\quad - \frac{1}{16}\,\theta^{\rho\sigma}\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}(\partial_\mu A^a_\nu)A^a_\beta\gamma^\beta\not{\partial}\,.
\end{aligned}
$$

Finally, the operator T_3 containing three fields is

$$
T_3 = \begin{pmatrix} R & r_{12} & r_{13} \\ r_{21} & 0 & r_{23} \\ r_{31} & -r_{23} & 0 \end{pmatrix}, \tag{A.4}
$$

with

$$
\begin{aligned}
R^{fe,\mu\nu} &= -\frac{3}{16}\,\theta^{\rho\sigma}\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\epsilon^{afe}A^a_\alpha\bar\psi\gamma^\beta\psi \\
r^a_{12} &= -\frac{3}{32}\,\theta^{\rho\sigma}\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\epsilon^{abc}A^b_\mu A^c_\alpha\bar\psi_2\gamma^\beta\not{\partial} \\
r^a_{13} &= \frac{3}{32}\,\theta^{\rho\sigma}\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\epsilon^{abc}A^b_\mu A^c_\alpha\bar\psi_1\gamma^\beta\not{\partial} \\
r^a_{21} &= \frac{3i}{16}\theta^{\rho\sigma}\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\epsilon^{abc}\gamma^\beta\psi_2 A^b_\mu A^c_\alpha \\
r^a_{31} &= -\frac{3i}{16}\,\theta^{\rho\sigma}\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}\epsilon^{abc}\gamma^\beta\psi_1 A^b_\mu A^c_\alpha \\
r_{23} &= -\frac{3}{16}\,\theta^{\rho\sigma}\Delta^{\mu\nu\alpha}_{\rho\sigma\beta}A^1_\mu A^2_\nu A^3_\alpha\gamma^\beta\not{\partial}\,.
\end{aligned}
$$

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